

# A new smoothness result for Caputo-type fractional ordinary differential equations <sup>\*</sup>

Binjie Li <sup>†</sup>, Xiaoping Xie <sup>‡</sup>, Shiquan Zhang <sup>§</sup>

School of Mathematics, Sichuan University, Chengdu 610064, China

## Abstract

We present a new smoothness result for Caputo-type fractional ordinary differential equations, which reveals that, subtracting a non-smooth function that can be obtained by the information available, a non-smooth solution belongs to  $C^m$  for some positive integer  $m$ .

**Keywords:** Caputo, fractional differential equation, smoothness.

## 1 Introduction

Let us consider the following model problem: seek  $0 < h \leq a$  and

$$y \in \left\{ v \in C[0, h] : \|v - c_0\|_{C[0, h]} \leq b \right\}$$

such that

$$\begin{cases} D_*^\alpha y = f(x, y), & 0 \leq x \leq h, \\ y(0) = c_0, \end{cases} \quad (1.1)$$

where  $a > 0$ ,  $b > 0$ ,  $0 < \alpha < 1$ ,  $c_0 \in \mathbb{R}$ , and

$$f \in C([0, a] \times [c_0 - b, c_0 + b]).$$

Above, the Caputo-type fractional differential operator  $D_*^\alpha : C[0, h] \rightarrow C_0^\infty(0, h)'$  is given by

$$D_*^\alpha z := D J^{1-\alpha}(z - z(0)) \quad (1.2)$$

for all  $z \in C[0, h]$ , where  $D$  denotes the well-known first order generalized differential operator, and the Riemann-Liouville fractional integral operator  $J^{1-\alpha} : C[0, h] \rightarrow C[0, h]$  is defined by

$$J^{1-\alpha} z(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} z(t) dt, \quad 0 \leq x \leq h,$$

for all  $z \in C[0, h]$ .

By [2, Lemma 2.1], the above problem is equivalent to seeking solutions of the following Volterra integration equation:

$$y(x) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(t)) dt. \quad (1.3)$$

Diethelm and Ford [2] proved that, if  $f$  is continuous, then (1.3) has a solution  $y \in C[0, h]$  for some  $0 < h \leq a$ , and this solution is unique if  $f$  is Lipschitz continuous. A natural question arises whether  $y$  can be smoother than being continuous. This is not only of theoretical value, but also of great importance in developing numerical methods for (1.3).

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<sup>†</sup>Email: libinjiefem@yahoo.com

<sup>‡</sup>Corresponding author. Email: xpxie@scu.edu.cn

<sup>§</sup>Email: shiquanzhang@scu.edu.cn

To this question, Miller and Feldstein [5] gave the first answer: if  $f$  is analytic, then  $y$  is analytic in  $(0, h)$  for some  $0 < h \leq a$ . Then Lubich [4] considered the behavior of the solution near 0. He showed that, if  $f$  is analytic at the origin, then there exists a function  $Y$  of two variables that is analytic at the origin such that

$$y(x) = Y(x, x^\alpha), \quad 0 \leq x \leq h,$$

for some  $0 < h \leq a$ . The above work suggests that non-smoothness of the solution to (1.1) is generally unavoidable. However, Diethelm [1] established a sufficient and necessary condition under which  $y$  is analytic on  $[0, h]$  for some  $0 < h \leq a$ . But, since we have already seen that non-smoothness of  $y$  is generally unavoidable, it is not surprising that this condition is unrealistic. Recently, Deng [3] proposed two conditions: under the first condition the solution belongs to  $C^m$  for some positive integer  $m$ ; under the second one the solution is a polynomial. It should be noted that, the second condition is just the one proposed in [1], and the first condition is also unrealistic.

The main result of this paper is that, although the solution  $y$  of (1.1) does not generally belong to  $C^m$  for some positive integer  $m$ , we can still construct a non-smooth function of the form

$$S(x) := c_0 + \sum_{j=1}^n c_j x^{\gamma_j},$$

such that

$$y - S \in C^m,$$

provided  $f$  is sufficiently smooth. Most importantly, given  $c_0$  and  $f$ , we can obtain  $S$  by a simple computation. This is significant in the development of numerical methods for (1.1). In addition, we obtain a sufficient and necessary condition under which  $y \in C^m$ . We note that this condition is essentially the same as the first condition mentioned already in [3, Theorem 2.8], but the necessity was not considered therein.

The rest of this paper is organized as follows. In Section 2 we introduce some basic notation and preliminaries. In Section 3 we state the main results of this paper, and present their proofs in Section 4.

## 2 Notation and Preliminaries

Let  $0 < h < \infty$ . We use  $C[0, h]$  to denote the space of all continuous real functions defined on  $[0, h]$ . For any  $k \in \mathbb{N}_{>0}$  and  $0 \leq \gamma \leq 1$ , define

$$C^k[0, h] := \left\{ v \in C[0, h] : v^{(j)} \in C[0, h] \text{ for } j = 1, 2, \dots, k \right\}, \quad (2.1)$$

$$C^{k, \gamma}[0, h] := \left\{ v \in C^k[0, h] : \max_{0 \leq x < y \leq h} |v|_{C^{k, \gamma}[0, h]} < \infty \right\}, \quad (2.2)$$

and endow the above two spaces with two norms respectively by

$$\|v\|_{C^k[0, h]} := \max_{0 \leq j \leq k} \max_{0 \leq x \leq h} |v^{(j)}(x)| \quad \text{for all } v \in C^k[0, h], \quad (2.3)$$

$$\|v\|_{C^{k, \gamma}[0, h]} := \max \left\{ \|v\|_{C^k[0, h]}, |v|_{C^{k, \gamma}[0, h]} \right\} \quad \text{for all } v \in C^{k, \gamma}[0, h]. \quad (2.4)$$

Here the semi-norm  $| \cdot |_{C^{k, \gamma}[0, h]}$  is given by

$$|v|_{C^{k, \gamma}[0, h]} := \sup_{0 \leq x < y \leq h} \frac{|v^{(k)}(x) - v^{(k)}(y)|}{(y - x)^\gamma}$$

for all  $v \in C^{k, \gamma}[0, h]$ , and it is obvious that  $C^k[0, h]$  coincides with  $C^{k, 0}[0, h]$ .

For any  $s \in \mathbb{N}_{>0}$ , define

$$\Lambda_s := \left\{ \beta = (\beta_1, \beta_2, \dots, \beta_s) \in \{1, 2\}^s \right\},$$

and, for any  $\beta \in \Lambda_s$ , we use the following notation:

$$\partial_\beta g := \frac{\partial}{\partial x_{\beta_s}} \frac{\partial}{\partial x_{\beta_{s-1}}} \cdots \frac{\partial}{\partial x_{\beta_1}} g(x_1, x_2),$$

where  $g$  is a real function of two variables. In addition, we define

$$\Lambda_0 := \{\emptyset\},$$

and denote by  $\partial_\emptyset$  the identity mapping.

### 3 Main Results

Let us first make the following assumption on  $f$ .

**Assumption 1.** *There exist a positive integer  $n$ , and a positive constant  $M$  such that*

$$f \in C^n([0, a] \times [c_0 - b, c_0 + b]),$$

$$\max_{(x, y) \in [0, a] \times [c_0 - b, c_0 + b]} \max_{\substack{0 \leq i \leq n \\ 0 \leq j \leq n \\ i+j \leq n}} \left| \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial y^j} f(x, y) \right| \leq M.$$

Throughout this paper, we assume that the above assumption is fulfilled.

Define  $J \in \mathbb{N}$  and a strictly increasing sequence  $\{\gamma_i\}_{i=1}^J$  by

$$\{\gamma_j : 1 \leq j \leq J\} = \{i + j\alpha : i, j \in \mathbb{N}, 0 < i + j\alpha < m\}, \quad (3.1)$$

where

$$m := \max \{j \in \mathbb{N} : j < n\alpha\}. \quad (3.2)$$

Define  $c_1, c_2, \dots, c_J \in \mathbb{R}$  by

$$Q(x) - S(x) + c_0 \in \text{span} \{x^{i+j\alpha} : i, j \in \mathbb{N}, i + j\alpha \geq m\}, \quad (3.3)$$

where

$$Q(x) := \sum_{s=0}^{n-1} \sum_{\beta \in \Lambda_s} \frac{\partial_\beta f(0, c_0)}{\Gamma(\alpha)} \int_0^x (x - t_0)^{\alpha-1} dt_0 \prod_{k=1}^s \int_0^{t_{k-1}} \frac{1 + (-1)^{\beta_k+1}}{2} + \frac{1 + (-1)^{\beta_k}}{2} \sum_{j=1}^J \gamma_j c_j t_k^{\gamma_j-1} dt_k, \quad (3.4)$$

and

$$S(x) := c_0 + \sum_{j=1}^J c_j x^{\gamma_j}. \quad (3.5)$$

Above and throughout, a product of a sequence of integrals should be understood in expanded form. For example, (3.4) is understood by

$$Q(x) := \sum_{s=0}^{n-1} \sum_{\beta \in \Lambda_s} \frac{\partial_\beta f(0, c_0)}{\Gamma(\alpha)} \int_0^x (x - t_0)^{\alpha-1} dt_0 \int_0^{t_0} \frac{1 + (-1)^{\beta_1+1}}{2} + \frac{1 + (-1)^{\beta_1}}{2} \sum_{j=1}^J \gamma_j c_j t_1^{\gamma_j-1} dt_1$$

$$\int_0^{t_1} \frac{1 + (-1)^{\beta_2+1}}{2} + \frac{1 + (-1)^{\beta_2}}{2} \sum_{j=1}^J \gamma_j c_j t_2^{\gamma_j-1} dt_2$$

$$\dots$$

$$\int_0^{t_{s-1}} \frac{1 + (-1)^{\beta_s+1}}{2} + \frac{1 + (-1)^{\beta_s}}{2} \sum_{j=1}^J \gamma_j c_j t_s^{\gamma_j-1} dt_s.$$

**Remark 3.1.** *It is easy to see that we can express  $Q$  in the form*

$$Q(x) = \sum_{j=1}^L d_j x^{\gamma_j},$$

where  $\{\gamma_j\}_{j=1}^L$  is a strictly increasing sequence such that  $\gamma_J < \gamma_{J+1}$  and

$$\{\gamma_j : 1 \leq j \leq L\} = \{i + j\alpha : i, j \in \mathbb{N}, i \leq n-1, 1 \leq j \leq 1 + (n-1)\gamma_J\}.$$

Moreover, for  $1 \leq j \leq J$ , the value of  $d_j$  only depends on  $c_0, c_1, \dots, c_{j-1}$ , and  $f$  (more precisely,  $\partial_\beta f(0, c_0)$ ,  $\beta \in \Lambda_s$ ,  $1 \leq s \leq n-1$ ). Obviously, there exist(s) uniquely  $c_1, c_2, \dots, c_J$  such that (3.3) holds, and hence  $c_1, c_2, \dots, c_J$  are/is well-defined. Furthermore, if  $\gamma_J + \alpha - m > 0$ , then

$$Q - S \in C^{m, \gamma_J + \alpha - m}[0, a];$$

and if  $\gamma_J + \alpha - m = 0$ , then

$$Q - S \in C^{m, \alpha}[0, a].$$

**Remark 3.2.** Note that,  $S$  only depends on  $c_0$  and

$$\{\partial_\beta f(0, c_0) : \beta \in \Lambda_s, 0 \leq s < n\}.$$

Since  $c_0$  and  $f$  are already available, we can obtain  $S$  by a simple calculation.

Define

$$h^* := \min \left\{ a, \left( \frac{b\Gamma(1+\alpha)}{M} \right)^{\frac{1}{\alpha}} \right\}.$$

By [2, Theorem 2.2] we know that there exists a unique solution  $y^* \in C[0, h^*]$  to (1.1). Now we state the most important result of this paper in the following theorem.

**Theorem 3.1.** There exist two positive constant  $C_0$  and  $C_1$  that only depends on  $a, \alpha$  and  $M$ , such that, for any  $0 < h \leq h^*$  and  $K > 0$  such that

$$\|(Q - S)'\|_{C^{m-1}[0, h]} + C_1 h^\alpha + C_0 h^\alpha \sum_{j=1}^m K^j \leq K,$$

we have  $y^* - S \in C^m[0, h]$  and

$$\|(y^* - S)'\|_{C^{m-1}[0, h]} \leq K. \quad (3.6)$$

**Corollary 3.1.** There exists  $0 < h \leq h^*$  such that  $y^* \in C^m[0, h]$  if, and only if,

$$\frac{\partial^i}{\partial x^i} f(0, c_0) = 0 \quad \text{for all } 0 \leq i < m. \quad (3.7)$$

**Remark 3.3.** Corollary 3.1 states that  $y^* \in C^1[0, h]$  for some  $0 < h \leq h^*$  if and only if  $f(0, c_0) = 0$ . So we only have  $y^* \in C[0, h] \setminus C^1[0, h]$ , if  $f(0, c_0) \neq 0$ . This yields great difficulty in developing high order numerical methods for (1.1), although  $y^* \in C^m(0, h]$ . Many numerical methods for (1.1) may not even converge theoretically, since they require that  $y^* \in C^m[0, h]$  for some positive integer  $m$ . However, we can obtain the numerical values of  $y^*$  at some left-most nodes by solving the following problem ( $y^* = y + S$ ): seek  $y \in C^m[0, \tilde{h}]$  such that

$$y(x) = c_0 - S(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(t) + S(t)) dt, \quad 0 \leq x \leq \tilde{h},$$

where  $\tilde{h} \ll h$ . Then we start the numerical methods for (1.1).

**Remark 3.4.** Assuming that  $f$  satisfies  $f(x, c_0) = 0$  for all  $0 \leq x \leq a$ , it is easy to see that

$$c_i = 0 \quad \text{for all } 1 \leq i \leq J,$$

and hence  $S = c_0$ . Then Theorem 3.1 implies  $y^* \in C^m[0, h]$ . Actually, in this case, it is easy to see that  $y^* = c_0$ .

**Remark 3.5.** Put

$$\Theta := \{1 \leq j \leq J : \gamma_j \notin \mathbb{N}\}.$$

Obviously,

$$\sum_{j \in \Theta} c_j x^{\gamma_j}$$

is the singular part (compared to the  $C^m$  regularity) in  $S$ , and thus the singular part in  $y^*$ . Corollary 3.1 essentially claims that (3.7) holds if and only if  $c_j = 0$  for all  $j \in \Theta$ . Since (3.7) is rare, we can consider singularity as an intrinsic property of solutions to fractional differential equations. In addition, we have the following result: that  $c_j = 0$  for all  $1 \leq j \leq J$  is equivalent to that  $c_j = 0$  for all  $j \in \Theta$ . This is contained in the proof of Corollary 3.1 in Section 4.3.

## 4 Proofs

Let  $0 < h < \infty$ . For any  $k \in \mathbb{N}$  and  $\gamma \in [0, 1]$ , define

$$\mathcal{C}^{k,\gamma}[0, h] := \left\{ v \in C^{k,\gamma}[0, h] : v^{(j)}(0) = 0, \quad j = 0, 1, 2, \dots, k \right\}, \quad (4.1)$$

$$\widehat{\mathcal{C}}^{k,\gamma}[0, h] := \left\{ v \in \mathcal{C}^{k,\gamma}[0, h] : \|v + S - c_0\|_{C[0, h]} \leq b \right\}. \quad (4.2)$$

In particular, we use  $\mathcal{C}^k[0, h]$  and  $\widehat{\mathcal{C}}^k[0, h]$  to abbreviate  $\mathcal{C}^{k,0}[0, h]$  and  $\widehat{\mathcal{C}}^{k,0}[0, h]$  respectively for  $k \in \mathbb{N}_{>0}$ , and use  $\mathcal{C}[0, h]$  and  $\widehat{\mathcal{C}}[0, h]$  to abbreviate  $\mathcal{C}^0[0, h]$  and  $\widehat{\mathcal{C}}^0[0, h]$  respectively. In addition, for a function  $v$  defined on  $(0, h]$  with  $h > 0$ , by  $v \in \mathcal{C}^{k,\gamma}[0, h]$  we mean that, setting  $v(0) := 0$ , the function  $v$  belongs to  $\mathcal{C}^{k,\gamma}[0, h]$ .

In the remainder of this paper, unless otherwise specified, we use  $C$  to denote a positive constant that only depends on  $\alpha$ ,  $a$  and  $M$ , and its value may differ at each occurrence. By the definitions of  $c_1$ ,  $c_2$ ,  $\dots$ ,  $c_J$ , it is easy to see that  $|c_j| \leq C$  for all  $1 \leq j \leq J$ , and we use this implicitly in the forthcoming analysis.

### 4.1 Some Auxiliary Results

We start by introducing some operators. For  $0 < h \leq a$ , define  $\mathcal{P}_{1,h} : \widehat{\mathcal{C}}^m[0, h] \rightarrow \mathcal{C}[0, h]$ ,  $\mathcal{P}_{2,h} : \widehat{\mathcal{C}}^m[0, h] \rightarrow \mathcal{C}[0, h]$ , and  $\mathcal{P}_{3,h} : \widehat{\mathcal{C}}^m[0, h] \rightarrow \mathcal{C}[0, h]$ , respectively, by

$$\mathcal{P}_{1,h}z(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \mathcal{G}_{1,h}z(t) dt, \quad (4.3)$$

$$\mathcal{P}_{2,h}z(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \mathcal{G}_{2,h}z(t) dt, \quad (4.4)$$

$$\mathcal{P}_{3,h}z(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \mathcal{G}_{3,h}z(t) dt, \quad (4.5)$$

for all  $z \in \widehat{\mathcal{C}}^m[0, h]$ , where  $\mathcal{G}_{1,h}z$ ,  $\mathcal{G}_{2,h}z$ ,  $\mathcal{G}_{3,h}z \in \mathcal{C}[0, h]$  are given respectively by

$$\begin{aligned} \mathcal{G}_{1,h}z(t_0) := & \sum_{s=1}^n \sum_{\substack{\beta \in \Lambda_s \\ \beta_s=2}} \prod_{k=1}^{s-1} \int_0^{t_{k-1}} \frac{1 + (-1)^{\beta_k+1}}{2} + \frac{1 + (-1)^{\beta_k}}{2} \sum_{j=1}^J \gamma_j c_j t_k^{\gamma_j-1} dt_k \\ & \int_0^{t_{s-1}} z'(t_s) \partial_\beta f(t_s, z(t_s) + S(t_s)) dt_s, \end{aligned} \quad (4.6)$$

$$\begin{aligned} \mathcal{G}_{2,h}z(t_0) := & \sum_{\substack{\beta \in \Lambda_n \\ \beta_n=2}} \prod_{k=1}^{n-1} \int_0^{t_{k-1}} \frac{1 + (-1)^{\beta_k+1}}{2} + \frac{1 + (-1)^{\beta_k}}{2} \sum_{j=1}^J \gamma_j c_j t_k^{\gamma_j-1} dt_k \\ & \int_0^{t_{n-1}} \partial_\beta f(t_n, z(t_n) + S(t_n)) \sum_{j=1}^J \gamma_j c_j t_n^{\gamma_j-1} dt_n, \end{aligned} \quad (4.7)$$

$$\begin{aligned} \mathcal{G}_{3,h}z(t_0) := & \sum_{\substack{\beta \in \Lambda_n \\ \beta_n=1}} \prod_{k=1}^{n-1} \int_0^{t_{k-1}} \frac{1 + (-1)^{\beta_k+1}}{2} + \frac{1 + (-1)^{\beta_k}}{2} \sum_{j=1}^J \gamma_j c_j t_k^{\gamma_j-1} dt_k \\ & \int_0^{t_{n-1}} \partial_\beta f(t_n, z(t_n) + S(t_n)) dt_n, \end{aligned} \quad (4.8)$$

for all  $0 \leq t_0 \leq h$ .

Then let us present the following important results for the above operators.

**Lemma 4.1.** *Let  $0 < h \leq a$ . For any  $z \in \widehat{\mathcal{C}}^m[0, h]$ , we have*

$$\frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, z(t) + S(t)) dt = Q(x) + \mathcal{P}_{1,h}z(x) + \mathcal{P}_{2,h}z(x) + \mathcal{P}_{3,h}z(x) \quad (4.9)$$

for all  $0 \leq x \leq h$ .

*Proof.* Let  $\beta \in \Lambda_s$  with  $1 \leq s < n$ . For any  $0 < t_s \leq h$ , applying the fundamental theorem of calculus yields

$$\begin{aligned} \partial_\beta f(t_s, z(t_s) + S(t_s)) &= \partial_\beta f(\epsilon, z(\epsilon) + S(\epsilon)) + \int_\epsilon^{t_s} \partial_{\tilde{\beta}} f(t_{s+1}, z(t_{s+1}) + S(t_{s+1})) dt_{s+1} + \\ &\quad \int_\epsilon^{t_s} \left( z'(t_{s+1}) + \sum_{j=1}^J \gamma_j c_j t_{s+1}^{\gamma_j-1} \right) \partial_{\tilde{\beta}} f(t_{s+1}, z(t_{s+1}) + S(t_{s+1})) dt_{s+1} \end{aligned}$$

for all  $0 < \epsilon \leq t_s$ , where  $\tilde{\beta} := (\beta_1, \beta_2, \dots, \beta_s, 1)$  and  $\tilde{\tilde{\beta}} := (\beta_1, \beta_2, \dots, \beta_s, 2)$ . Taking limits on both sides of the above equation as  $\epsilon$  approaches  $0+$ , we obtain

$$\begin{aligned} \partial_\beta f(t_s, z(t_s) + S(t_s)) &= \partial_\beta f(0, c_0) + \int_0^{t_s} \partial_{\tilde{\beta}} f(t_{s+1}, z(t_{s+1}) + S(t_{s+1})) dt_{s+1} + \\ &\quad \int_0^{t_s} \left( z'(t_{s+1}) + \sum_{j=1}^J \gamma_j c_j t_{s+1}^{\gamma_j-1} \right) \partial_{\tilde{\beta}} f(t_{s+1}, z(t_{s+1}) + S(t_{s+1})) dt_{s+1}. \end{aligned}$$

Using this equality repeatedly, we easily obtain (4.9). This completes the proof.  $\blacksquare$

**Lemma 4.2.** Let  $0 < h \leq a$ . For any  $z \in \hat{\mathcal{C}}^m[0, h]$ , we have  $\mathcal{P}_{1,h}z \in \mathcal{C}^{m,\alpha}[0, h]$  and

$$\|(\mathcal{P}_{1,h}z)'\|_{C^{m-1}[0,h]} \leq Ch^\alpha \sum_{j=1}^m \|z'\|_{C^{m-1}[0,h]}^j, \quad (4.10)$$

$$\left| (\mathcal{P}_{1,h}z)^{(m)} \right|_{C^{0,\alpha}[0,h]} \leq C \sum_{j=1}^m \|z'\|_{C^{m-1}[0,h]}^j. \quad (4.11)$$

**Lemma 4.3.** Let  $0 < h \leq a$ . For any  $z \in \hat{\mathcal{C}}^m[0, h]$ , we have  $\mathcal{P}_{2,h}z, \mathcal{P}_{3,h}z \in \mathcal{C}^{m,\alpha}[0, h]$  and

$$\|(\mathcal{P}_{2,h}z)'\|_{C^{m-1}[0,h]} + \|(\mathcal{P}_{3,h}z)'\|_{C^{m-1}[0,h]} \leq Ch^\alpha, \quad (4.12)$$

$$\left| (\mathcal{P}_{2,h}z)^{(m)} \right|_{C^{0,\alpha}[0,h]} + \left| (\mathcal{P}_{3,h}z)^{(m)} \right|_{C^{0,\alpha}[0,h]} \leq C. \quad (4.13)$$

To prove the above two lemmas, we need several lemmas below.

**Lemma 4.4.** Let  $0 < h \leq a$  and  $g \in \mathcal{C}^m[0, h]$ . We have  $w \in \mathcal{C}^{m,\alpha}[0, h]$  and

$$\|w'\|_{C^{m-1}[0,h]} \leq Ch^\alpha \|g'\|_{C^{m-1}[0,h]}, \quad (4.14)$$

$$\left| w^{(m)} \right|_{C^{0,\alpha}[0,h]} \leq C \|g^{(m)}\|_{C[0,h]}, \quad (4.15)$$

where

$$w(x) := \int_0^x (x-t)^{\alpha-1} g(t) dt, \quad 0 \leq x \leq h.$$

*Proof.* Since  $g \in \mathcal{C}^m[0, h]$  we have

$$w^{(i)}(x) = \int_0^x (x-t)^{\alpha-1} g^{(i)}(t) dt, \quad 1 \leq i \leq m.$$

Then  $w \in \mathcal{C}^m[0, h]$  and (4.14) follow, and (4.15) follows from [6, Theorem 3.1]. This completes the proof.  $\blacksquare$

**Lemma 4.5.** Let  $0 < h \leq a$ , and  $k, l \in \mathbb{N}$  such that  $k \leq m$  and  $l\alpha \leq 1$ . For any  $g \in \mathcal{C}^{k,l\alpha}[0, h]$ , define

$$w(x) := \int_0^x \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} g(t) dt, \quad 0 < x \leq h.$$

Then we have the following results:

- If  $(l+1)\alpha \leq 1$ , then we have  $w \in \mathcal{C}^{k,(l+1)\alpha}[0, a]$  and

$$\|w\|_{\mathcal{C}^{k,(l+1)\alpha}} \leq C \|g\|_{\mathcal{C}^{k,l\alpha}}.$$

- If  $(l+1)\alpha > 1$ , then we have  $w \in \mathcal{C}^{k+1,(l+1)\alpha-1}[0, a]$  and

$$\|w\|_{\mathcal{C}^{k+1,(l+1)\alpha-1}} \leq C \|g\|_{\mathcal{C}^{k,l\alpha}}.$$

For any  $0 < h \leq a$ ,  $w \in \mathcal{C}[0, h]$ , and  $\beta \in \Lambda_s$  with  $1 \leq s \leq n$ , define  $\mathcal{T}_{w,\beta,h} : \widehat{\mathcal{C}}^m[0, h] \rightarrow \mathcal{C}[0, h]$  by

$$\mathcal{T}_{w,\beta,h} z(x) := w(x) \partial_\beta f(x, z(x) + S(x)),$$

for all  $z \in \widehat{\mathcal{C}}^m[0, h]$ .

**Lemma 4.6.** For  $0 \leq k \leq m$ , we have  $\mathcal{T}_{w,\beta,h} z \in \mathcal{C}^{\min\{k,n-s\}}[0, h]$  and

$$\|\mathcal{T}_{w,\beta,h} z\|_{\mathcal{C}^{\min\{k,n-s\}}[0,h]} \leq C \|w\|_{\mathcal{C}^k[0,h]} \sum_{j=0}^{\min\{k,n-s\}} \|z'\|_{\mathcal{C}^{m-1}[0,h]}^j \quad (4.16)$$

for all  $0 < h \leq a$ ,  $w \in \mathcal{C}^k[0, h]$ ,  $\beta \in \Lambda_s$  with  $1 \leq s \leq n$ , and  $z \in \widehat{\mathcal{C}}^m[0, h]$ .

The proofs of Lemmas 4.5 and 4.6 are presented in Appendix A. In the rest of this subsection, we give the proofs of Lemmas 4.2 and 4.3.

**Proof of Lemma 4.2.** By (4.3), (4.6), and Lemma 4.4, it suffices to show that, for each  $\beta \in \Lambda_s$  with  $\beta_s = 2$ , we have  $g_0 \in \mathcal{C}^m[0, h]$  and

$$\|g_0\|_{\mathcal{C}^m[0,h]} \leq C \sum_{j=1}^{\min\{m,n-s+1\}} \|z'\|_{\mathcal{C}^{m-1}[0,h]}^j, \quad (4.17)$$

where, if  $s = 1$ , then

$$g_0(x) := \int_0^x z'(t) \partial_2 f(t, z(t) + S(t)) dt;$$

if  $2 \leq s \leq n$ , then

$$\begin{aligned} g_0(x) &:= \int_0^x \left( \frac{1 + (-1)^{\beta_1+1}}{2} + \frac{1 + (-1)^{\beta_1}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} \right) g_1(t) dt, \\ g_1(x) &:= \int_0^x \left( \frac{1 + (-1)^{\beta_2+1}}{2} + \frac{1 + (-1)^{\beta_2}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} \right) g_2(t) dt, \\ &\vdots \\ g_{s-2}(x) &:= \int_0^x \left( \frac{1 + (-1)^{\beta_{s-1}+1}}{2} + \frac{1 + (-1)^{\beta_{s-1}}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} \right) g_{s-1}(t) dt, \\ g_{s-1}(x) &:= \int_0^x z'(t) \partial_\beta f(t, z(t) + S(t)) dt. \end{aligned}$$

To do so, we proceed as follows. If  $s = 1$ , then by Lemma 4.6 we obtain  $g_0 \in \mathcal{C}^m[0, h]$  and (4.17). Let us suppose that  $2 \leq s \leq n$ . By Lemma 4.6 it follows  $g_{s-1} \in \mathcal{C}^{\min\{m,n-s+1\}}[0, h]$  and

$$\|g_{s-1}\|_{\mathcal{C}^{\min\{m,n-s+1\}}[0,h]} \leq C \sum_{j=1}^{\min\{m,n-s+1\}} \|z'\|_{\mathcal{C}^{m-1}[0,h]}^j.$$

Then, by the simple estimate

$$(n - s + 1) + (s - 1)\alpha > m,$$

applying Lemma 4.5 to  $g_{s-2}, g_{s-3}, \dots, g_0$  successively yields  $g_0 \in \mathcal{C}^m[0, h]$  and (4.17). This completes the proof of Lemma 4.2.  $\blacksquare$

**Proof of Lemma 4.3.** Let us first show that  $\mathcal{G}_{2,h}z \in \mathcal{C}^m[0, h]$  and

$$\|\mathcal{G}_{2,h}z\|_{\mathcal{C}^m[0,h]} \leq C. \quad (4.18)$$

By (4.7) it suffices to show that, for any  $\beta \in \Lambda_n$  with  $\beta_n = 2$ , we have  $g_0 \in \mathcal{C}^m[0, h]$  and

$$\|g_0\|_{\mathcal{C}^m[0,h]} \leq C, \quad (4.19)$$

where

$$\begin{aligned} g_0(x) &:= \int_0^x \left( \frac{1 + (-1)^{\beta_1+1}}{2} + \frac{1 + (-1)^{\beta_1}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} \right) g_1(t) dt, \\ g_1(x) &:= \int_0^x \left( \frac{1 + (-1)^{\beta_2+1}}{2} + \frac{1 + (-1)^{\beta_2}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} \right) g_2(t) dt, \\ &\vdots \\ g_{n-2}(x) &:= \int_0^x \left( \frac{1 + (-1)^{\beta_{s-1}+1}}{2} + \frac{1 + (-1)^{\beta_{s-1}}}{2} \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} \right) g_{s-1}(t) dt, \\ g_{n-1}(x) &:= \int_0^x \partial_\beta f(t, z(t) + S(t)) \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} dt, \end{aligned}$$

for all  $0 \leq x \leq h$ . Noting the fact that

$$\partial_\beta f(\cdot, z(\cdot) + S(\cdot)) \in \mathcal{C}[0, h],$$

and  $\gamma_j \geq \alpha$  for all  $1 \leq j \leq J$ , we easily obtain  $g_{n-1} \in \mathcal{C}^{0,\alpha}[0, h]$  and

$$\|g_{n-1}\|_{\mathcal{C}^{0,\alpha}[0,h]} \leq C.$$

Then, applying Lemma 4.5 to  $g_{n-2}, g_{n-3}, \dots, g_0$  successively, and using the fact  $n\alpha > m$ , we obtain  $g_0 \in \mathcal{C}^m[0, h]$  and (4.19). Thus we have showed  $\mathcal{G}_{2,h}z \in \mathcal{C}^m[0, h]$  and (4.18).

Similarly, we can show that  $\mathcal{G}_{3,h}z \in \mathcal{C}^m[0, h]$  and  $\|\mathcal{G}_{3,h}z\|_{\mathcal{C}^m[0,h]} \leq C$ . Consequently, by (4.4), (4.5), and Lemma 4.4, we infer that  $\mathcal{P}_{2,h}z, \mathcal{P}_{3,h}z \in \mathcal{C}^{m,\alpha}[0, h]$ , and (4.12) and (4.13) hold. This completes the proof.  $\blacksquare$

## 4.2 Proof of Theorem 3.1

By Lemmas 4.2 and 4.3 there exist two positive constants  $C_0$  and  $C_1$  that only depend on  $a, \alpha$  and  $M$ , such that

$$\|(\mathcal{P}_{1,h}z)'\|_{\mathcal{C}^{m-1}[0,h]} \leq C_0 h^\alpha \sum_{j=1}^m \|z'\|_{\mathcal{C}^{m-1}[0,h]}^j, \quad (4.20)$$

$$\|(\mathcal{P}_{2,h}z)'\|_{\mathcal{C}^{m-1}[0,h]} + \|(\mathcal{P}_{3,h}z)'\|_{\mathcal{C}^{m-1}[0,h]} \leq C_1 h^\alpha, \quad (4.21)$$

for all  $0 < h \leq a$  and  $z \in \widehat{\mathcal{C}}^m[0, h]$ . Let  $0 < h \leq h^*$  and  $K > 0$  such that

$$\|(Q - S)'\|_{\mathcal{C}^{m-1}[0,h]} + C_1 h^\alpha + C_0 h^\alpha \sum_{j=1}^m K^j \leq K. \quad (4.22)$$

Define  $\mathcal{J} : V \rightarrow \mathcal{C}[0, h]$  by

$$\mathcal{J}z(x) := c_0 - S(x) + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, z(t) + S(t)) dt, \quad (4.23)$$

for all  $z \in V$  and  $x \in [0, h]$ , where

$$V := \left\{ v \in \widehat{\mathcal{C}}^m[0, h] : \|v'\|_{\mathcal{C}^{m-1}[0,h]} \leq K \right\}. \quad (4.24)$$



**Remark 4.1.** It is clear that  $V$  is a bounded, closed, convex subset of  $C^m[0, h]$ .

**Remark 4.2.** Let  $\delta > 0$ . If we put

$$K := \|(Q - S)'\|_{C^{m-1}[0, h]} + C_1 a^\alpha + \delta,$$

$$h := \min \left\{ h^*, \left( \delta^{-1} C_0 \sum_{j=1}^m K^j \right)^{-\frac{1}{\alpha}} \right\},$$

then (4.22) holds.

For the operator  $\mathcal{J}$ , we have the following key result.

**Lemma 4.7.** For each  $z \in V$ , we have  $\mathcal{J}z \in V$  and

$$\left| (\mathcal{J}z)^{(m)} \right|_{C^{0, \gamma}[0, h]} \leq \left| (Q - S)^{(m)} \right|_{C^{0, \gamma}[0, h]} + C \sum_{j=0}^m K^j, \quad (4.25)$$

where  $\gamma := \alpha$  if  $\gamma_J + \alpha = m$ , and  $\gamma := \gamma_J + \alpha - m$  if  $\gamma_J + \alpha > m$ .

*Proof.* Let us first show  $\mathcal{J}z \in V$ . Using (4.23) and the fact  $h \leq \left( \frac{b\Gamma(1+\alpha)}{M} \right)^{\frac{1}{\alpha}}$ , we have

$$|\mathcal{J}z(x) + S(x) - c_0| = \frac{1}{\Gamma(\alpha)} \left| \int_0^x (x-t)^{\alpha-1} f(t, z(t) + S(t)) dt \right| \leq \frac{Mh^\alpha}{\Gamma(1+\alpha)} \leq b$$

for all  $x \in [0, h]$ , and so

$$\|\mathcal{J}z + S - c_0\|_{C[0, h]} \leq b.$$

By Lemma 4.1 we have

$$\mathcal{J}z(x) = c_0 - S(x) + Q(x) + \mathcal{P}_{1, h}z(x) + \mathcal{P}_{2, h}z(x) + \mathcal{P}_{3, h}z(x), \quad (4.26)$$

and then, by Lemmas 4.2 and 4.3, and the fact  $c_0 - S + Q \in C^m[0, h]$ , we obtain  $\mathcal{J}z \in C^m[0, h]$ . It remains, therefore, to show that

$$\|(\mathcal{J}z)'\|_{C^{m-1}[0, h]} \leq K. \quad (4.27)$$

To this end, note that, by (4.26), (4.20) and (4.21) we obtain

$$\|(\mathcal{J}z)'\|_{C^{m-1}[0, h]} \leq \|(Q - S)'\|_{C^{m-1}[0, h]} + C_1 h^\alpha + C_0 h^\alpha \sum_{j=1}^m K^j,$$

and then (4.27) follows from (4.22). We have thus showed  $\mathcal{J}z \in V$ .

Finally, let us show (4.25). By Lemmas 4.2 and 4.3 we obtain

$$\left| (\mathcal{P}_{1, h}z)^{(m)} \right|_{C^{0, \alpha}[0, h]} + \left| (\mathcal{P}_{2, h}z)^{(m)} \right|_{C^{0, \alpha}[0, h]} + \left| (\mathcal{P}_{3, h}z)^{(m)} \right|_{C^{0, \alpha}[0, h]} \leq C \sum_{j=0}^m \|z'\|_{C^{m-1}[0, h]}^j \leq C \sum_{j=0}^m K^j.$$

From the fact  $\gamma \leq \alpha$  it follows

$$\left| (\mathcal{P}_{1, h}z)^{(m)} \right|_{C^{0, \gamma}[0, h]} + \left| (\mathcal{P}_{2, h}z)^{(m)} \right|_{C^{0, \gamma}[0, h]} + \left| (\mathcal{P}_{3, h}z)^{(m)} \right|_{C^{0, \gamma}[0, h]} \leq C \sum_{j=0}^m K^j.$$

Using this estimate and the fact that  $(Q - S)^{(m)} \in C^{0, \gamma}$  by the definitions of  $Q$  and  $S$ , the desired estimate (4.25) follows from (4.26). This completes the proof.  $\blacksquare$

By the famous Arzelà-Ascoli Theorem and Lemma 4.7, it is evident that  $\mathcal{J} : V \rightarrow V$  is a compact operator, where  $V$  is endowed with norm  $\|\cdot\|_{C^m[0, h]}$ . Therefore, since  $V$  is a bounded, closed, convex subset of  $C^m[0, h]$ , using the Schauder Fixed-Point Theorem gives that there exists  $z \in V$  such that

$$\mathcal{J}z = z.$$

Putting

$$y(x) := z(x) + S(x), \quad 0 \leq x \leq h,$$

we obtain

$$y(x) = c_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t, y(t)) dt, \quad 0 \leq x \leq h.$$

By [2, Lemma 2.1], the above  $y$  is a solution of (1.1), and then, since  $y^*$  is the unique solution of (1.1) on  $[0, h^*]$ , we have  $y^* = y$  on  $[0, h]$ . Therefore, it is obvious that  $y^* - S \in C^m[0, h]$  and (3.6) hold. This completes the proof of Theorem 3.1.

### 4.3 Proof of Corollary 3.1

Let us first state the following fact. For each  $1 \leq j \leq J$ , by the definition of  $c_j$ , a straightforward computing yields

$$c_j = \sum_{t \in \Upsilon_{j,1} \cup \Upsilon_{j,2}} t, \quad (4.28)$$

where

$$\Upsilon_{j,1} := \bigcup_{\substack{1 \leq s < n \\ s+\alpha=\gamma_j}} \left\{ \frac{B(\alpha, 1+s) \partial_1^s f(0, c_0)}{\Gamma(\alpha)} \right\}, \quad (4.29)$$

$$\Upsilon_{j,2} := \bigcup_{s=1}^{n-1} \bigcup_{k=1}^s \bigcup_{\substack{\beta \in \Lambda_s \\ \#\beta=k \\ \Gamma_\beta \neq \emptyset}} \left\{ \frac{B\left(\alpha, 1+s-k+\sum_{l=1}^k \gamma_{i_l}\right) \partial_\beta f(0, c_0)}{\Gamma(\alpha) \prod_{l=1}^k \gamma_{i_l}} \prod_{l=1}^k c_{i_l} \gamma_{i_l} : (i_1, i_2, \dots, i_k) \in \Xi_{\beta,j} \right\}. \quad (4.30)$$

Above,  $B(\cdot, \cdot)$  denotes the standard beta function, and

$$\begin{aligned} \#\beta &:= \sum_{\substack{1 \leq i \leq s \\ \beta_i=2}} 1, \\ \Xi_{\beta,j} &:= \left\{ (i_1, i_2, \dots, i_{\#\beta}) : \alpha + s - \#\beta + \sum_{j=1}^{\#\beta} \gamma_{i_j} = \gamma_j \right\}, \end{aligned}$$

for all  $1 \leq s < n$  and  $\beta \in \Lambda_s$ .

To prove Corollary 3.1, by Theorem 3.1 it suffices to show that (3.7) is equivalent to

$$c_j = 0 \quad \text{for all } j \in \Theta, \quad (4.31)$$

where

$$\Theta := \{1 \leq j \leq J : \gamma_j \notin \mathbb{N}\}.$$

But, by (4.28), (4.29) and (4.30), an obvious induction gives

$$c_j = 0 \quad \text{for all } 1 \leq j \leq J, \quad (4.32)$$

if (3.7) holds. Therefore, it remains to show that (4.31) implies (3.7).

To this end, let us assume that (4.31) holds. Note that we have (4.32). If this statement was false, then let

$$j_0 := \min \{1 \leq j \leq J : c_j \neq 0\}.$$

Obviously, we have  $j_0 > 1$  and  $\gamma_{j_0} \in \mathbb{N}$ , and in this case,  $\Upsilon_{j_0,1}$  is empty. Thus, by (4.28) we have

$$c_{j_0} = \sum_{t \in \Upsilon_{j_0,2}} t.$$

But, by the definition of  $\Upsilon_{j_0,2}$  and the fact that  $c_j = 0$  for all  $1 \leq j < j_0$ , it is straightforward that  $c_{j_0} = 0$ , which is contrary to the definition of  $j_0$ . Therefore (4.32) holds indeed. Using this result, from (4.28) and (4.30) it follows

$$c_j = \sum_{t \in \Upsilon_{j,1}} t \quad \text{for all } 1 \leq j \leq J,$$

and then, using (4.32) again, we obtain (3.7). This completes the proof of Corollary 3.1.

## Appendix A Proofs of Lemmas 4.5 and 4.6

To prove Lemma 4.5, we need the following two lemmas.

**Lemma A.1.** *Let  $h > 0$ ,  $\gamma > 0$  and  $g \in \mathcal{C}^1[0, h]$ . We have  $w \in \mathcal{C}^1[0, h]$  and*

$$w'(x) = \int_0^x t^{\gamma-1} g'(t) dt. \quad (\text{A.1})$$

where

$$w(x) := \int_0^x t^{\gamma-1} g(t) dt, \quad 0 \leq x \leq h.$$

Since the proof of this lemma is straightforward, it is omitted.

**Lemma A.2.** *Let  $0 < h \leq a$ , and  $l \in \mathbb{N}_{>0}$  such that  $l\alpha \leq 1 < (l+1)\alpha$ . For any  $g \in \mathcal{C}^{0,l\alpha}[0, h]$ , we have  $w \in \mathcal{C}^{0,(l+1)\alpha-1}[0, h]$  and*

$$\|w\|_{\mathcal{C}^{0,(l+1)\alpha-1}[0,h]} \leq C \|g\|_{\mathcal{C}^{0,l\alpha}[0,h]},$$

where

$$w(x) := \sum_{j=1}^J \gamma_j c_j x^{\gamma_j-1} g(x), \quad 0 < x \leq h.$$

*Proof.* It suffices to prove that, for any  $1 \leq j \leq J$ , we have  $v \in \mathcal{C}[0, h]$  and

$$\|v\|_{\mathcal{C}^{0,(l+1)\alpha-1}[0,h]} \leq C \|g\|_{\mathcal{C}^{0,l\alpha}[0,h]},$$

where  $v(x) := x^{\gamma_j-1} g(x)$ ,  $0 < x \leq h$ . Noting the fact that  $l\alpha + \gamma_j > 1$  and  $g \in \mathcal{C}^{0,l\alpha}[0, h]$ , we easily obtain  $v \in \mathcal{C}[0, h]$  and

$$\|v\|_{\mathcal{C}[0,h]} \leq C \|g\|_{\mathcal{C}^{0,l\alpha}[0,h]}.$$

It remains, therefore, to prove that

$$|v(y) - v(x)| \leq C(y-x)^{(l+1)\alpha-1} \|g\|_{\mathcal{C}^{0,l\alpha}[0,h]}$$

for all  $0 < x < y \leq h$ . Moreover, since it holds

$$\begin{aligned} |v(y) - v(x)| &= |y^{\gamma_j-1} g(y) - x^{\gamma_j-1} g(x)| \\ &= |y^{\gamma_j-1} (g(y) - g(x)) + (y^{\gamma_j-1} - x^{\gamma_j-1}) g(x)| \\ &\leq (y^{\gamma_j-1} (y-x)^{l\alpha} + |y^{\gamma_j-1} - x^{\gamma_j-1}| x^{l\alpha}) \|g\|_{\mathcal{C}^{0,l\alpha}[0,h]}, \end{aligned}$$

by the fact  $g \in \mathcal{C}^{0,l\alpha}[0, h]$ , we only need to prove that

$$y^{\gamma_j-1} (y-x)^{l\alpha} + |y^{\gamma_j-1} - x^{\gamma_j-1}| x^{l\alpha} \leq C(y-x)^{(l+1)\alpha-1} \quad (\text{A.2})$$

for all  $0 < x < y \leq h$ .

Let us first consider the case of  $\gamma_j < 1$ . A simple algebraic calculation gives

$$(x^{\gamma_j-1} - y^{\gamma_j-1}) x^{l\alpha} = (y-x)^{l\alpha+\gamma_j-1} (A^{\gamma_j-1} - (1+A)^{\gamma_j-1}) A^{l\alpha},$$

where  $A := \frac{x}{y-x}$ . If  $0 \leq A \leq 1$ , then by the fact  $l\alpha + \gamma_j - 1 > 0$  we have

$$(A^{\gamma_j-1} - (1+A)^{\gamma_j-1}) A^{l\alpha} < A^{l\alpha+\gamma_j-1} \leq 1.$$

If  $A > 1$ , then using the Mean Value Theorem and the fact  $l\alpha + \gamma_j - 2 < 0$  gives

$$(A^{\gamma_j-1} - (1+A)^{\gamma_j-1}) A^{l\alpha} < (1-\gamma_j) A^{l\alpha+\gamma_j-2} < (1-\gamma_j) < 1.$$

Consequently, we obtain

$$(x^{\gamma_j-1} - y^{\gamma_j-1}) x^{l\alpha} < (y-x)^{l\alpha+\gamma_j-1},$$

which, together with the trivial estimate

$$y^{\gamma_j-1}(y-x)^{l\alpha} < (y-x)^{\gamma_j-1}(y-x)^{l\alpha} = (y-x)^{l\alpha+\gamma_j-1},$$

yields (A.2).

Then, since (A.2) is evident in the case of  $\gamma_j = 1$ , let us consider the case of  $1 < \gamma_j < 2$ . Since  $0 < \gamma_j - 1 < 1$ , we have

$$y^{\gamma_j-1} - x^{\gamma_j-1} < (y-x)^{\gamma_j-1}.$$

By the definition of  $\gamma_j$  it is clear that

$$\gamma_j - 1 \geq (l+1)\alpha - 1.$$

Using the above two estimates, we obtain

$$|y^{\gamma_j-1} - x^{\gamma_j-1}| x^{l\alpha} \leq C(y^{\gamma_j-1} - x^{\gamma_j-1}) \leq C(y-x)^{(l+1)\alpha-1},$$

which, together with the estimate

$$y^{\gamma_j-1}(y-x)^{l\alpha} \leq C(y-x)^{l\alpha} \leq C(y-x)^{(l+1)\alpha-1},$$

indicates (A.2).

Finally, let us consider the case of  $\gamma_j \geq 2$ . Using the Mean Value Theorem gives

$$|y^{\gamma_j-1} - x^{\gamma_j-1}| x^{l\alpha} \leq C(y-x)^{(l+1)\alpha-1}.$$

and then, by the obvious estimate

$$y^{\gamma_j-1}(y-x)^{l\alpha} \leq C(y-x)^{(l+1)\alpha-1},$$

we obtain (A.2). This completes the proof. ■

**Proof of Lemma 4.5** Since  $g \in \mathcal{C}^{k,l\alpha}[0, h]$ , by Lemma A.1 we have  $w \in \mathcal{C}^k[0, h]$  and

$$w^{(i)}(x) = \int_0^x \sum_{j=1}^J \gamma_j c_j t^{\gamma_j-1} g^{(i)}(t) dt, \quad i = 0, 1, 2, \dots, k. \quad (\text{A.3})$$

It follows

$$\|w\|_{\mathcal{C}^k[0, h]} \leq C \|g\|_{\mathcal{C}^k[0, h]}.$$

Therefore, it remains to prove that

$$\left| w^{(k)} \right|_{\mathcal{C}^{0, (l+1)\alpha}[0, h]} \leq C \|g\|_{\mathcal{C}^{k, l\alpha}[0, h]} \quad (\text{A.4})$$

if  $(l+1)\alpha \leq 1$ ; and that  $w^{(k+1)} \in \mathcal{C}^{0, (l+1)\alpha-1}[0, h]$  and

$$\left\| w^{(k+1)} \right\|_{\mathcal{C}^{0, (l+1)\alpha-1}[0, h]} \leq C \|g\|_{\mathcal{C}^{k, l\alpha}[0, h]} \quad (\text{A.5})$$

if  $(l+1)\alpha > 1$ .

Let us first consider (A.4). Noting the fact that  $g^{(k)} \in \mathcal{C}^{0, l\alpha}[0, h]$  and  $\gamma_j \geq \alpha$  for all  $1 \leq j \leq J$ , by (A.3) a simple computing gives that

$$\left| w^{(k)}(y) - w^{(k)}(x) \right| \leq C \left| g^{(k)} \right|_{\mathcal{C}^{0, l\alpha}[0, h]} (y-x)^{(l+1)\alpha}$$

for all  $0 \leq x < y \leq h$ , which implies (A.4).

Then let us consider (A.5). Since  $g^{(k)} \in \mathcal{C}^{0, l\alpha}$ , by Lemma A.2 we have  $v \in \mathcal{C}^{0, (l+1)\alpha-1}[0, h]$  and

$$\|v\|_{\mathcal{C}^{0, (l+1)\alpha-1}[0, h]} \leq C \left\| g^{(k)} \right\|_{\mathcal{C}^{0, l\alpha}[0, h]},$$

where

$$v(x) := \sum_{j=1}^J \gamma_j c_j x^{\gamma_j-1} g^{(k)}(x), \quad 0 < x \leq h.$$

Then, by (A.3) we readily obtain  $w^{(k+1)} \in \mathcal{C}^{0, (l+1)\alpha-1}$  and (A.5), and thus complete the proof of this lemma. ■

Before proving Lemma 4.6, let us introduce the following lemma.

**Lemma A.3.** Let  $0 < h \leq a$  and  $\gamma > 0$ . For any  $g \in \mathcal{C}^k[0, h]$  with  $1 \leq k \leq m$ , we have  $w \in \mathcal{C}^{k-1}[0, h]$  and

$$\|w\|_{\mathcal{C}^{k-1}[0, h]} \leq C \|g^{(k)}\|_{\mathcal{C}[0, h]},$$

where

$$w(x) := g(x)x^{\gamma-1}, \quad 0 < x \leq h,$$

and  $C$  is a positive constant that only depends on  $a$ ,  $k$  and  $\gamma$ .

*Proof.* If  $k = 1$ , then, by the Mean Value Theorem and the fact  $g(0) = 0$ , this lemma is evident. Thus, below we assume that  $2 \leq k \leq m$ . In the rest of this proof, for ease of notation, the symbol  $C$  denotes a positive constant that only depends on  $a$ ,  $k$  and  $\gamma$ , and its value may differ at each occurrence.

Let us first show that, for  $0 \leq i < k$ , we have  $w_i \in \mathcal{C}[0, h]$  and

$$\|w_i\|_{\mathcal{C}[0, h]} \leq C \|g^{(k)}\|_{\mathcal{C}[0, h]}, \quad (\text{A.6})$$

where

$$w_i(x) := w^{(i)}(x), \quad 0 < x \leq h.$$

To this end, let  $0 \leq i < k$ , and note that an elementary computing gives

$$w_i(x) = \sum_{j=0}^i c_{ij} g^{(j)}(x) x^{\gamma-1-i+j}, \quad 0 < x \leq h, \quad (\text{A.7})$$

where  $c_{ij}$  is a constant that only depends on  $\gamma$ ,  $i$  and  $j$ , for all  $0 \leq j \leq i$ . Since  $g \in \mathcal{C}^k[0, h]$ , we have  $g^{(j)} \in \mathcal{C}^{k-j}[0, h]$ , and then, applying Taylor's formula with integral remainder yields

$$g^{(j)}(x) = \frac{1}{(k-j-1)!} \int_0^x (x-t)^{k-j-1} g^{(k)}(t) dt, \quad 0 \leq x \leq h.$$

It follows that

$$\left| g^{(j)}(x) x^{\gamma-1-i+j} \right| \leq \frac{\|g^{(k)}\|_{\mathcal{C}[0, h]}}{(k-j)!} x^{\gamma+k-(i+1)}, \quad 0 < x \leq h. \quad (\text{A.8})$$

Since  $\gamma + k - (i+1) \geq \gamma > 0$ , this implies  $g^{(j)}(x) x^{\gamma-1-i+j} \in \mathcal{C}[0, h]$  and

$$\left\| g^{(j)}(\cdot) (\cdot)^{\gamma-i-1+j} \right\|_{\mathcal{C}[0, h]} \leq C \|g^{(k)}\|_{\mathcal{C}[0, h]}.$$

Therefore, by (A.7) it follows  $w_i \in \mathcal{C}[0, h]$  and (A.6).

Then let us proceed to prove this lemma. Let  $i < k-1$ . Note that by (A.7) we have

$$w'_i(x) = w_{i+1}(x), \quad 0 < x \leq h.$$

Since we have already proved that  $w_i, w_{i+1} \in \mathcal{C}[0, h]$ , by the Mean Value Theorem it is evident that  $w_i \in \mathcal{C}^1[0, h]$  and

$$w'_i(x) = w_{i+1}(x), \quad 0 \leq x \leq h.$$

It follows  $w_0 \in \mathcal{C}^{k-1}[0, h]$  and

$$w_0^{(i)} = w_i, \quad 0 \leq i < k,$$

and hence, by (A.6) we have

$$\|w_0\|_{\mathcal{C}^{k-1}[0, h]} \leq C \|g^{(k)}\|_{\mathcal{C}[0, h]}.$$

Noting the fact  $w = w_0$ , this completes the proof. ■

**Proof of Lemma 4.6** Below we employ the well-known principle of mathematical induction to prove this lemma. Firstly, it is clear that (4.16) holds in the case  $k = 0$ . Secondly, assuming that (4.16) holds for  $k = l$  where  $0 \leq l < m-1$ , let us prove that (4.16) holds for  $k = l+1$ . To this end, a straightforward computing gives

$$(\mathcal{T}_{w, \beta, h} z)'(x) = \mathcal{T}_{w', \beta, h} z(x) + \mathcal{T}_{w, \tilde{\beta}, h} z(x) + \mathcal{T}_{\tilde{w}, \tilde{\beta}, h} z(x) \quad (\text{A.9})$$

for all  $0 < x \leq h$ , where  $\tilde{\beta} := (\beta_1, \beta_2, \dots, \beta_s, 1)$ ,  $\tilde{\tilde{\beta}} := (\beta_1, \beta_2, \dots, \beta_s, 2)$ , and

$$\tilde{w}(x) := w(x) \left( z'(x) + \sum_{j=1}^J \gamma_j c_j x^{\gamma_j-1} \right).$$

Since  $w \in \mathcal{C}^k[0, h]$ , we have  $w' \in \mathcal{C}^{k-1}[0, h]$ , and by Lemma A.3 we have  $\tilde{w} \in \mathcal{C}^{k-1}[0, h]$ ; consequently,  $\mathcal{T}_{w', \beta, h} z$  and  $\mathcal{T}_{\tilde{w}, \tilde{\beta}, h} z$  are well-defined, and they both belong to  $\mathcal{C}[0, h]$ . Therefore, by the Mean Value Theorem, and the fact  $\mathcal{T}_{w, \beta, h} z \in \mathcal{C}[0, h]$ , it follows that  $\mathcal{T}_{w, \beta, h} z \in \mathcal{C}^1[0, h]$ , and (A.9) holds for all  $0 \leq x \leq h$ . By our assumption, we have the following results:  $\mathcal{T}_{w', \beta, h} z \in \mathcal{C}^{\min\{k-1, n-s\}}[0, h]$  and

$$\|\mathcal{T}_{w', \beta, h} z\|_{\mathcal{C}^{\min\{k-1, n-s\}}[0, h]} \leq C \|w'\|_{\mathcal{C}^{k-1}[0, h]} \sum_{j=0}^{\min\{k-1, n-s\}} \|z'\|_{\mathcal{C}^{m-1}[0, h]}^j;$$

$\mathcal{T}_{w, \tilde{\beta}, h} z \in \mathcal{C}^{\min\{k-1, n-s-1\}}[0, h]$  and

$$\left\| \mathcal{T}_{w, \tilde{\beta}, h} z \right\|_{\mathcal{C}^{\min\{k-1, n-s-1\}}[0, h]} \leq C \|w\|_{\mathcal{C}^{k-1}[0, h]} \sum_{j=0}^{\min\{k-1, n-s-1\}} \|z'\|_{\mathcal{C}^{m-1}[0, h]}^j;$$

$\mathcal{T}_{\tilde{w}, \tilde{\tilde{\beta}}, h} z \in \mathcal{C}^{\min\{k-1, n-s-1\}}[0, h]$  and

$$\left\| \mathcal{T}_{\tilde{w}, \tilde{\tilde{\beta}}, h} z \right\|_{\mathcal{C}^{\min\{k-1, n-s-1\}}[0, h]} \leq C \|\tilde{w}\|_{\mathcal{C}^{k-1}[0, h]} \sum_{j=0}^{\min\{k-1, n-s-1\}} \|z'\|_{\mathcal{C}^{m-1}[0, h]}^j.$$

In addition, by Lemma A.3 we easily obtain

$$\|\tilde{w}\|_{\mathcal{C}^{k-1}[0, h]} \leq C \|w\|_{\mathcal{C}^k[0, h]} \left( 1 + \|z'\|_{\mathcal{C}^{k-1}[0, h]} \right).$$

As a consequence, we obtain  $\mathcal{T}_{w, \beta, h} z \in \mathcal{C}^{\min\{k, n-s\}}$  and

$$\|(\mathcal{T}_{w, \beta, h} z)'\|_{\mathcal{C}^{\min\{k-1, n-s-1\}}[0, h]} \leq C \|w\|_{\mathcal{C}^k[0, h]} \sum_{j=0}^{\min\{k, n-s\}} \|z'\|_{\mathcal{C}^{m-1}[0, h]}^j.$$

Then (4.16) follows from the obvious estimate

$$\|\mathcal{T}_{w, \beta, h} z\|_{\mathcal{C}[0, h]} \leq C \|w\|_{\mathcal{C}[0, h]}.$$

This completes the proof of Lemma 4.6. ■

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